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FORMAL MATRICES AND RINGS CLOSE TO REGULAR

A. N. Abyzov and A. A. Tuganbaev

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ABSTRACT. This paper contains new and known results on formal matrix rings close to regular. The main results are given with proofs.

1. Preliminaries

All rings are assumed to be associative and with nonzero identity element; all modules are assumed to be unitary. Let $R_1, R_2, \dots, R_n$ be rings and let M_{ij} be (R_i, R_j)-bimodules such that $M_{ii} = R_i$ for all $1 \le i, j \le n$. In addition, let $\varphi_{ijk}\colon M_{ij} \otimes_{R_j} M_{jk} \to M_{ik}$ be (R_i, R_k)-bimodule homomorphisms such that φ_{iij} and φ_{ijj} are canonical isomorphisms for all $1 \le i, j \le n$. We set $a \circ b = \varphi_{ijk}(a \otimes b)$ for $a \in M_{ij}$ and $b \in M_{jk}$. We denote by K the set of all $n \times n$ matrices (m_{ij}) with elements $m_{ij} \in M_{ij}$ for all $1 \le i, j \le n$. It is easy to verify that K is a ring with respect to ordinary operations of addition and of multiplication if and only if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a \in M_{ik}$, $b \in M_{kl}$, and $c \in M_{lj}$, $1 \le i, k, l, j \le n$. The obtained ring K is called a *formal matrix ring* of order n; it is denoted by $K(\{M_{ij}\}\colon \{\varphi_{ikj}\})$. If

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

are formal matrix rings of order 2, then the ordered family $(R, S, M, N, \varphi, \psi)$ is called a *Morita context* or a *pre-equivalence situation*.

The formal matrix ring $K(\{M_{ij}\}\colon \{\varphi_{ikj}\})$ of order n, in which $M_{ij} = R$ for all $1 \le i, j \le n$, is called a *formal matrix ring over R of order n*; it is denoted by $K_n(R)$ or $K_n(R\colon \{\varphi_{ikj}\})$. For a formal matrix ring $K_n(R\colon \{\varphi_{ijk}\})$ over R of order n, we set $\eta_{ijk} = \varphi_{ijk}(1 \otimes 1)$ for all $1 \le i, j, k \le n$. Then $a \circ b = \varphi_{ijk}(a \otimes b) = \eta_{ijk} ab$ for all $a, b \in R$. For every $a \in R$, we have $a\eta_{ijk} = \varphi_{ijk}(a \otimes 1) = \varphi_{ijk}(1 \otimes a) = \eta_{ijk} a$. Therefore, $\eta_{ijk} \in C(R)$, and the following conditions hold:

(1) $\eta_{iij} = \eta_{ijj} = 1$, $1 \le i, j \le n$;
(2) $\eta_{ijk}\eta_{ikl} = \eta_{ijl}\eta_{jkl}$, $1 \le i, j, k, l \le n$.

The first condition holds, since φ_{iij} and φ_{ijj} are canonical isomorphisms. Since the operation $\circ$ is associative, we have $\eta_{ijk}\,\eta_{ikl}\,abc = \eta_{ijl}\,\eta_{jkl}\,abc$ for all $a, b, c \in R$. By setting $a = b = c = 1$, we obtain the second condition. For every family $\{\eta_{ijk} \mid 1 \le i, j, k \le n\}$ of central elements of R satisfying the first condition and the second condition, we can set $\varphi_{ijk}(a \otimes b) = \eta_{ijk} ab$ for all $a, b \in R$. It is directly verified that $K_n(R\colon \{\varphi_{ikj}\})$ is a formal matrix ring over R of order n. Therefore, the formal matrix ring $K_n(R\colon \{\varphi_{ikj}\})$ is uniquely defined by the family $\{\eta_{ijk} \mid 1 \le i, j, k \le n\}$ of central elements. In this case, the formal matrix ring $K_n(R\colon \{\varphi_{ikj}\})$ is denoted by $K_n(R\colon \{\eta_{ikj}\})$.

Let R be a ring and let $\beta_1, \dots, \beta_n \in C(R)$ with $n \ge 2$. We define η_{ijk} for all $1 \le i, j, k \le n$ by the relation

$$\eta_{ijk} = \begin{cases} 1 & \text{if } i = j \text{ or } j = k, \\ \beta_j & \text{if } i,\ j,\ k \text{ are distinct}, \\ \beta_i\beta_j & \text{if } i = k \ne j. \end{cases}$$

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